Sequence and Series of functions

Def: Sequence of real valued functions :

let f_n be real valued function defined on $E \subseteq R$ for each $n \in N$. Then seq $\{f_1, f_2, f_3, \dots, f_n, \dots\}$ is called seq. of real valued functions on E. It is denoted by $\{f_n : E \to R, n \in N\}$ or simply by $\{f_n\}$ or $\langle f_n \rangle$.

eg: let f_n is a real valued function defined by $f_n(x) = x^n$ $x \in [0,1]$, then $\{f_1(x), f_2(x), f_3(x), \dots\} = \{x, x^2, x^3, \dots\}$ is a seq of real valued functions defined on $[0,1]$.

Def: Series of real valued functions : let $\{f_n\}$ be seq of real valued functions defined on set $E \subseteq R$, then expression $f_1 + f_2 + \dots + f_n + \dots = \sum_{n=1}^{\infty} f_n$ is called series of real valued functions defined on E.

eg let $\{f_n\}$ be seq of real valued function defined on $[0,1]$ by $f_n(x) = \frac{\cos nx}{n^2}$, $x \in [0,1]$, then

$$\sum f_n(x) = f_1(x) + f_2(x) + \dots + f_n(x) + \dots$$
$$= \cos x + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \dots + \frac{\cos nx}{n^2} + \dots$$

is called series of real valued function on $[0,1]$.

Pointwise and Uniform Convergence of seq of functions

Let $\{f_n\}$ be seq of real valued function defined on $E \subseteq R$. Then for each $x_i \in E$, $\{f_n(x_i)\} = \{f_1(x_i), f_2(x_i), \dots\}$ be seq of real numbers.

let the seq of numbers $\{f_n(x_i)\}$ converges to real no. $f(x_i)$ (say). In this way sequences $\{f_n(x_1)\}, \{f_n(x_2)\}, \{f_n(x_3)\}$ ---- at the points $x_1, x_2, x_3, \dots$ of E converge to $f(x_1)$, $f(x_2)$, $f(x_3)$, --- i.e. all sequences of numbers $\{f_n(x)\}$ converge $\forall x \in E$. Then we can define a function f with domain E and range $\{f(x_1), f(x_2), \dots\}$ such that

$$f(x) = \lim_{n\to\infty} f_n(x) \quad \forall x \in E.$$

Thus seq of functions $\{f_n\}$ defined on set E is said to be pointwise convergent if for each $x \in E$, seq $\{f_n(x)\}$ of real no. converges.

In this case, we say $\{f_n\}$ converges to f pointwise on E and f is called pointwise limit function of seq $\{f_n\}$.

If sequence of functions $\{f_n\}$ converges pointwise to f, then for given $\epsilon > 0$, each $x_i \in E$, $\exists\ m_{x_i} \in N$ s.t

$$|f_n(x_i) - f(x_i)| < \epsilon \quad \forall\ n \geq m_{x_i}$$

In this way different points of E give rise to a seq $\{m_{x_i}\}$ of natural no. If this seq $\{m_{x_i}\}$ is bdded above and m is its lub i.e. $m = \text{lub}\{m_{x_i}\}$, then

$$|f_n(x) - f(x)| < \epsilon \quad \forall n \geq m, \ \forall x \in E.$$

In this case the integer m depends up on ϵ only and uniform for all $x \in E$.

So we say seq $\{f_n\}$ converges uniformly to f on E.

But if $\{m_{x_i}\}$ is not bdded above, then no such m exists and seq $\{f_n\}$ is not uniformly cgt on E.

Remarks

(1) It is clear that every uniformly cgt sequence is point wise cgt and uniform limit = point-wise limit. The difference between the point wise convergence and uniform convergence is as follows.

In case of pointwise convergence, $\forall \epsilon > 0$ and $x_i \in E$ $\exists$ +ve integer m_{x_i} (depending on x_i and ϵ both) where as in uniform convergence for each $\epsilon > 0$, $\exists$ an +ve integer m (depending on ϵ alone).

(2) Every point-wise convergent sequence is not uniformly convergent.

(3) If $\{f_n\}$ is not pointwise convergent on E, then $\{f_n\}$ can not uniformly convergent on E.

(4) If a seq is uniformly cgt, then uniform limit function is same as pointwise limit function.

Definition: Pointwise convergence of seq. of functions.

Let $\{f_n\}$ be seq. of real valued function defined on E. Then seq $\{f_n\}$ converges pointwise to a real valued function f defined on E iff for each $x \in E$ and given $\epsilon > 0$ $\exists$ $m \in N$ (depending upon ϵ and x) such that

$$|f_n(x) - f(x)| < \epsilon \quad \forall n \geq m.$$

We write it as $f_n \to f$ pointwise on E

or $\lim_{n \to \infty} f_n(x) = f(x) \quad \forall x \in E$

Definition: Uniform convergence of a seq. of functions.

A seq $\{f_n\}$ of functions defined on E is said to converge uniformly to a real valued function f defined on E iff given $\epsilon > 0$, $\exists$ $m \in N$ (independent of x but dependent on ϵ) such that

$$|f_n(x) - f(x)| < \epsilon \quad \forall n \geq m, \quad \forall x \in E.$$

We write it as $f_n \to f$ uniformly on E.

Pointwise and Uniform convergence of series of functions

Let ΣU_n be series of functions defined on E.

and define $f_1 = U_1$

$$f_2 = U_1 + U_2,$$

$$f_n = U_1 + U_2 + \cdots + U_n$$

Then seq. $\{f_n\}$ is seq of partial sums of series ΣU_n. Series ΣU_n converges pointwise to f on E iff seq $\{f_n\}$ converges pointwise to function f on E.

The limit function f of $\{f_n\}$ is called pointwise sum of ΣU_n and we write $\Sigma U_n(x) = f(x) \quad \forall x \in E$.

Ily ΣU_n converges uniformly on E iff seq $\{f_n\}$ of partial sum converges uniformly on E.

Definition: Uniformly bounded sequence of functions

A seq $\{f_n\}$ of real valued functions defined on set E is said to be uniformly bounded on E if $\exists$ +ve real no. K (independent of x and n) such that

$$|f_n(x)| < K \quad \forall n \in N \text{ and } \forall x \in E.$$

Here, no. K is called uniform bound of seq $\{f_n\}$ on E.

Definition: Uniformly bounded series of functions.

A series ΣU_n of real valued functions defined on E is said to be uniformly bounded on E iff $\exists$ the real no K (independent of x and n) such that

$$|f_n(x)| < K \quad \forall n \in N \text{ and } \forall x \in E$$

Where $f_n = U_1 + U_2 + \cdots + U_n$ be nth partial sum of ΣU_n.

No. K is called uniform bound of series ΣU_n.

Definition: Point of non-uniform convergence of seq of functions

A point at which the seq. does not converge uniformly in any neighbourhood of it, however small, is said to be a point of non uniform convergence.

Cauchy's Criterion for uniform Convergence of seq of functions

Th^m: State and prove Cauchy's criterion for uniform convergence of a seq. of functions.

Statement: A sequence of functions $\{f_n\}$ defined on E converges uniformly on E iff given $\epsilon > 0$ $\exists$ $m \in N$ s.t. $|f_{n+p}(x) - f_n(x)| < \epsilon$ $\forall$ $n \geq m$, $p \geq 1$ $\forall$ $x \in E$.

Proof:- First let seq $\{f_n\}$ converges uniformly to limit function f on E.

$\therefore$ given $\epsilon > 0$ $\exists$ $m \in N$ s.t.

$$|f_n(x) - f(x)| < \frac{\epsilon}{2} \quad \forall n \geq m, \ \forall x \in E \quad \text{---} (1)$$

Also from (1), for $n \geq m$, $p \geq 1$ and $\forall x \in E$

$$|f_{n+p}(x) - f(x)| < \frac{\epsilon}{2} \quad \text{---} (2)$$

$$\therefore |f_{n+p}(x) - f_n(x)| = |f_{n+p}(x) - f(x) + f(x) - f_n(x)|$$

$$\leq |f_{n+p}(x) - f(x)| + |f(x) - f_n(x)|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{by (1) and (2)}$$

$$\Rightarrow |f_{n+p}(x) - f_n(x)| < \epsilon \quad \forall n \geq m, \ p \geq 1, \ \forall x \in E \quad \text{---} (3)$$

Conversely let suppose that (3) holds for $\{f_n\}$.

Then for each $x \in E$, seq $\{f_n(x)\}$ is a Cauchy seq of real no., hence cgt as Cauchy seq of real no is cgt.

$\therefore$ $\exists$ a real no. y such that

$$\lim_{n\to\infty} f_n(x) = y.$$

$\therefore$ We can define a function $f: E \to R$ by

$$f(x) = y = \lim_{n\to\infty} f_n(x) \quad \forall x \in E. \qquad (4)$$

$\Rightarrow$ $f_n \to f$ point wise on E

We claim $f_n \to f$ uniformly on E.

from (3) keeping n fixed and making $p \to \infty$, we get

$$\lim_{p\to\infty} |f_{n+p}(x) - f_n(x)| < \epsilon \quad \forall n \geq m, \ \forall x \in E$$

$$\Rightarrow \left|\lim_{p\to\infty} f_{n+p}(x) - f_n(x)\right| < \epsilon \quad \forall n \geq m, \ \forall x \in E$$

$$\Rightarrow |f(x) - f_n(x)| < \epsilon \quad \forall n \geq m, \ \forall x \in E$$

$\Rightarrow$ $f_n \to f$ uniformly on E.

Thm: State and prove Cauchy's Criterion for uniform Convergence of Series of function.

Statement: A series Σu_n of real valued functions defined on E Converges uniformly on E iff given $\epsilon > 0$ $\exists$ $m \in N$ s.t.

$$|u_{n+1}(x) + u_{n+2}(x) + \cdots + u_{n+p}(x)| < \epsilon \quad \forall n \geq m, \ p \geq 1, \ \forall x \in E.$$

Proof: Let $f_n = u_1 + u_2 + \cdots + u_n$ be nth partial sum of Σu_n

Now Σu_n Converges uniformly on E

iff seq $\{f_n\}$ converges uniformly on E

iff given $\epsilon > 0$, $\exists$ $m \in N$ s.t

$$|f_{n+p}(x) - f_n(x)| < \epsilon \quad \forall n \geq m,\ p \geq 1 \quad \forall x \in E$$

{ By Cauchy's criterion for uniform convergence of seq of function }

iff $|U_{n+1}(x) + U_{n+2}(x) + \cdots + U_{n+p}(x)| < \epsilon \quad \forall n \geq m,\ p \geq 1 \quad \forall x \in E.$

which Competes the proof.

Q. Show that seq $\{f_n\}$ where $f_n(x) = \frac{nx}{1+n^2x^2}$, $x \in R$ is not uniformly cgt in any interval containing zero.

Sol^n Here $f_n(x) = \frac{nx}{1+n^2x^2}$, $x \in R$.

$$\therefore f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+n^2x^2} = \lim_{n\to\infty} \frac{\frac{x}{n}}{\frac{1}{n^2}+x^2}$$

$$= \frac{0}{0+x^2} = 0 \quad \forall x \in R - \{0\}.$$

Also when $x = 0$, $f_n(x) = 0 \quad \forall n \in N$

$$\Rightarrow f(x) = 0.$$

$$\therefore f(x) = \lim_{n\to\infty} f_n(x) = 0 \quad \forall x \in R.$$

$\therefore$ $\{f_n\}$ Converges point wise to 0 $\forall x \in R$

for uniform convergence

Let $\epsilon > 0$ be given, then

$$|f_n(x) - f(x)| = \left|\frac{nx}{1+n^2x^2} - 0\right| = \frac{n|x|}{1+n^2x^2} < \epsilon$$

if $n|x| < \epsilon + n^2x^2\epsilon$

if $\epsilon x^2 n^2 - |x| n + \epsilon > 0$

if $n > \frac{|x| + \sqrt{x^2 - 4\epsilon^2 x^2}}{2\epsilon x^2}$

if $n > \dfrac{1+\sqrt{1-4\epsilon^2}}{2\epsilon|x|}$.

When $x \to 0$ then $\dfrac{1+\sqrt{1-4\epsilon^2}}{2\epsilon|x|} \to \infty$

so that it is not possible to choose a the integer m such that

$$|f_n(x) - f(x)| < \epsilon \quad \forall\, n \geq m \quad \forall\, x \in R.$$

Hence $x=0$ is point of non uniform Convergece ~~of~~ for seq $\{f_n\}$.

Hence seq is not uniformly cgt on any interval $[a,b]$ conteining '0'.

Q: Show that the seq $\{f_n\}$ where $f_n(x) = \dfrac{x^n}{n}$, $x \in [0,1]$ converges uniformly to '0'.

Sol^n Here $f_n(x) = \dfrac{x^n}{n}$, $x \in [0,1]$

$$\therefore f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^n}{n} = 0 \quad \forall\, x \in [0,1]$$

$\therefore f_n \to f$ point wise on $[0,1]$. $\left[\because x \in [0,1] \Rightarrow x^n \in [0,1]\right]$

Let $\epsilon > 0$ be given.,

then $|f_n(x) - f(x)| = \left|\dfrac{x^n}{n} - 0\right| = \dfrac{x^n}{n} < \epsilon$

if $n > \dfrac{x^n}{\epsilon}$

Now max. value of $\dfrac{x^n}{\epsilon}$ on $[0,1]$ is $\dfrac{1}{\epsilon}$.

$\therefore$ Choose a the integer m s.t

$$m > \frac{1}{\epsilon} \geq \frac{x^n}{\epsilon} \quad \forall\, x \in [0,1]$$

then $|f_n(x) - f(x)| < \epsilon \quad \forall n \geq m$
$\forall x \in [0,1]$.

$\therefore f_n \to f$ uniformly on $[0,1]$.

i.e. $f_n \to 0$ uniformly on $[0,1]$

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(2) Show that seq $\{f_n\}$ where $f_n(x) = x^n$ is uniformly cgt on $[0,k]$, $k<1$ but only point wise cgt on $[0,1]$.

Sol^n Here $f_n(x) = x^n$, $x \in [0,1]$

$$\therefore f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

$\therefore f_n \to f$ point wise on $[0,1]$.

for uniform convergence, let $\epsilon > 0$ be given.

then for $0<x<1$

$|f_n(x) - f(x)| = |x^n - 0| = x^n < \epsilon$

if $\frac{1}{x^n} > \frac{1}{\epsilon}$

if $n \log\frac{1}{x} > \log\frac{1}{\epsilon}$

if $n > \dfrac{\log\frac{1}{\epsilon}}{\log\frac{1}{x}}$

$\left[\begin{array}{l} \text{Since } x \in (0,1) \\ \Rightarrow \frac{1}{x} > 1 \\ \Rightarrow \log\frac{1}{x} > 0 \end{array}\right]$

Now Max. value of $\dfrac{\log\frac{1}{\epsilon}}{\log\frac{1}{x}}$ on $(0,k]$, $k<1$

is $\dfrac{\log\frac{1}{\epsilon}}{\log\frac{1}{k}}$

Choose a +ve integer m, s.t

$$m > \frac{\log\frac{1}{\epsilon}}{\log\frac{1}{k}} \geq \frac{\log\frac{1}{\epsilon}}{\log\frac{1}{x}} \quad \forall x \in (0,k]$$

then $|f_n(x) - f(x)| < \epsilon \quad \forall\, n \geq m$
$\forall\, x \in [0, k],\ k < 1$

At $x = 0$, $|f_n(x) - f(x)| = |0 - 0| = 0 < \epsilon \quad \forall\, n \geq m$.

$\therefore$ given $\epsilon > 0$ $\exists$ $m \in \mathbb{N}$ s.t

$$|f_n(x) - f(x)| < \epsilon \quad \forall\, n \geq m, \ \forall\, x \in [0, k],\ k < 1$$

$\Rightarrow$ $\{f_n\}$ is uniformly cgt on $[0, k]$, $k < 1$.

Also when $x \to 1$, then $\dfrac{\log \frac{1}{\epsilon}}{\log \frac{1}{x}} \to \infty$

Thus it is not possible to find a the integer m s.t

$$|f_n(x) - f(x)| < \epsilon \quad \forall\, n \geq m, \ \forall\, x \in [0, 1].$$

Hence seq $\{f_n\}$ is not uniformly cgt on any interval containing 1 and in particular on $[0, 1]$.

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(3) Show that the seq $\{f_n\}$ where $f_n(x) = \dfrac{nx}{nx+1}$ is uniformly cgt on $[a, b]$, $a > 0$ but is only point wise cgt on $[0, b]$.

Soln: Here $f_n(x) = \dfrac{nx}{nx+1}$, $x \geq 0$

When $x = 0$, $f_n(x) = 0 \ \forall\, n \in \mathbb{N}$ $\quad \therefore f(x) = \lim_{n \to \infty} f_n(x) = 0$.

When $x > 0$, $f(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \dfrac{nx}{nx+1}$

$$= \lim_{n \to \infty} \frac{x}{x + \frac{1}{n}} = 1$$

$$\therefore f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$\therefore$ $f_n \to f$ point wise on $[0,b]$, $b>0$

for uniform convergence, let $\epsilon>0$ be given then for $x>0$, we have

$$|f_n(x) - f(x)| = \left|\frac{nx}{nx+1} - 1\right| = \left|\frac{-1}{nx+1}\right| = \frac{1}{nx+1} < \epsilon$$

if $nx+1 > \frac{1}{\epsilon}$

if $n > \frac{1}{x}\left(\frac{1}{\epsilon}-1\right)$

Now max. value of $\frac{1}{x}\left(\frac{1}{\epsilon}-1\right)$ on $[a,b]$, $a>0$ is $\frac{1}{a}\left(\frac{1}{\epsilon}-1\right)$

Choose a +ve integer m s.t

$$m > \frac{1}{a}\left(\frac{1}{\epsilon}-1\right) \geq \frac{1}{x}\left(\frac{1}{\epsilon}-1\right) \quad \forall x \in [a,b],\ a>0$$

then $|f_n(x) - f(x)| < \epsilon \quad \forall n \geq m,\ \forall x \in [a,b],\ a>0$

$\Rightarrow$ seq $\{f_n\}$ is uniformly cgt on $[a,b]$, $a>0$

Also when $x \to 0$, then $\frac{1}{x}\left(\frac{1}{\epsilon}-1\right) \to \infty$

$\therefore$ It is not possible to choose a +ve integer m s.t $\quad |f_n(x) - f(x)| < \epsilon \quad \forall n \geq m,\ \forall x \in [0,b]$.

Hence seq $\{f_n\}$ is not uniformly cgt on $[0,b]$, but is only pointwise cgt on $[0,b]$

Q: Show that seq. $\{f_n\}$ where $f_n(x) = e^{-nx}$ is uniformly cgt on $[a,b]$, $a>0$, but is only pointwise cgt on $[0,b]$.

Sol^n Here $f_n(x) = e^{-nx}$, $x \geq 0$ i.e. $x \in [0,b]$

$\therefore$ $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} e^{-nx} = \begin{cases} 1 & \text{if } x=0 \\ 0 & \text{if } x>0 \end{cases}$

$\therefore$ $f_n \to f$ pointwise on $[0,b]$.

For uniform convergence, let $\epsilon>0$ be given then for $x>0$, we have

$|f_n(x) - f(x)| = |e^{-nx} - 0| = e^{-nx} < \epsilon$

if $e^{nx} > \frac{1}{\epsilon}$

if $nx > \log\frac{1}{\epsilon}$

if $n > \frac{1}{x}\log\frac{1}{\epsilon}$

Now max value of $\frac{1}{x}\log\frac{1}{\epsilon}$ on $[a,b]$, $a\geq 0$ is $\frac{1}{a}\log\frac{1}{\epsilon}$

We can choose a +ve integer m s.t

$m > \frac{1}{a}\log\frac{1}{\epsilon} \geq \frac{1}{x}\log\frac{1}{\epsilon} \quad \forall x\in[a,b], a>0$

then $|f_n(x) - f(x)| < \epsilon \quad \forall n \geq m$
$\forall x \in [a,b], a>0.$

which shows that seq $\{f_n\}$ is uniformly cgt on $[a,b]$, $a>0$.

Also when $x\to 0$, then $\frac{1}{x}\log\frac{1}{\epsilon} \to \infty$

$\therefore$ It is impossible to choose a +ve integer m, s.t $|f_n(x) - f(x)| < \epsilon \quad \forall n \geq m, \forall x \in [0,b]$

Hence seq. $\{f_n\}$ is not uniformly cgt on $[0,1]$.

Q. Show that seq $\{f_n\}$ where $f_n(x) = \tan^{-1} nx$, $x \geq 0$ is uniformly cgt on $[a,1]$, $a>0$, but is only pt. wise cgt on $[0,1]$.

Sol^n: Here $f_n(x) = \tan^{-1} nx$, $x \geq 0$

$$\therefore f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \tan^{-1} nx = \begin{cases} \frac{\pi}{2}, & \text{if } x>0 \\ 0 & \text{if } x=0 \end{cases}$$

$\therefore$ $f_n \to f$ pointwise on $[0,1]$.

For uniform convergence, let $\epsilon>0$ be given then for $x>0$, we have

$$|f_n(x) - f(x)| = |\tan^{-1} nx - \tfrac{\pi}{2}| \qquad \left[\because \tan^{-1} nx + \cot^{-1} nx = \tfrac{\pi}{2}\right]$$
$$= |-\cot^{-1} nx|$$
$$= \cot^{-1} nx < \epsilon$$

if $nx > \cot \epsilon$

if $n > \dfrac{\cot\epsilon}{x}$

Now max. value of $\dfrac{\cot\epsilon}{x}$ on $[a,1]$, $a>0$ is $\dfrac{\cot\epsilon}{a}$

We can choose a the integer m s.t

$$m > \frac{\cot\epsilon}{a} \geq \frac{\cot\epsilon}{x} \quad \forall\, x\in[a,1],\ a>0$$

then $|f_n(x) - f(x)| < \epsilon$ $\quad \forall\, n \geq m$ and $\forall\, x\in[a,1], a>0$

$\therefore$ seq $\{f_n\}$ is uniformly cgt on $[a,1]$, $a>0$

Also when $x\to 0$, then $\dfrac{\cot\epsilon}{x} \to \infty$

$\therefore$ It is not possible to choose a the integer m s.t.

$|f_n(x) - f(x)| < \epsilon \quad \forall\, n \geq m$ and $\forall\, x \in [0,b]$

Hence $\{f_n\}$ is not uniformly cgt on $[0,b]$ but only pointwise cgt on $[0,b]$.

Q. Show that the series

$$\frac{1}{x+1} - \frac{1}{(x+1)(x+2)} - \frac{1}{(x+2)(x+3)} - \cdots - \frac{1}{[x+(n-1)](x+n)} - \cdots$$

Converges uniformly on $[0,1]$.

Sol^n Let given series is ΣU_n and $f_n = U_1 + U_2 + \cdots + U_n$ be nth partial sum of ΣU_n.

$$\therefore f_n(x) = U_1(x) + U_2(x) + \cdots + U_n(x)$$

$$= \frac{1}{x+1} - \frac{1}{(x+1)(x+2)} - \frac{1}{(x+2)(x+3)} - \cdots - \frac{1}{(x+\overline{n-1})(x+n)}$$

$$= \frac{1}{x+n} \qquad (\text{After solving})$$

$$\therefore f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{x+n} = 0 \quad \forall\, x \in [0,1]$$

$\therefore f_n \to f$ pointwise on $[0,1]$.

For uniform convergence, given $\epsilon > 0$, $x \in [0,1]$, we have

$$|f_n(x) - f(x)| = \left|\frac{1}{x+n} - 0\right| = \frac{1}{x+n} < \epsilon$$

if $x+n > \frac{1}{\epsilon}$

if $n > \frac{1}{\epsilon} - x$

Now max value of $\frac{1}{\epsilon} - x$ on $[0,1]$ is $\frac{1}{\epsilon}$

$\therefore$ We can choose a the integer m s.t

$$m > \frac{1}{\epsilon} \geq \frac{1}{\epsilon} - x \quad \forall\, x \in [0,1]$$

then $|f_n(x) - f(x)| < \epsilon \quad \forall\, n \geq m, \ \forall\, x \in [0,1]$

$\Rightarrow$ Seq. $\{f_n\}$ Converges uniformly on $[0,1]$

$\therefore$ $\sum U_n$ is uniformly cgt on $[0,1]$.

Q. Show that the series

$$\frac{x^2}{1+x} + \left(\frac{2x^2}{1+2x} - \frac{x^2}{1+x}\right) + \cdots + \left(\frac{nx^2}{1+nx} - \frac{(n-1)x^2}{1+(n-1)x}\right) + \cdots$$

Converges uniformly on $[0,1]$.

Sol^n Let given series is $\sum U_n$ and $f_n = U_1 + U_2 + \cdots + U_n$ be nth partial sum of $\sum U_n$.

$$\therefore f_n(x) = U_1(x) + U_2(x) + \cdots + U_n(x)$$

$$= \frac{x^2}{1+x} + \left(\frac{2x^2}{1+2x} - \frac{x^2}{1+x}\right) + \cdots + \left(\frac{nx^2}{1+nx} - \frac{(n-1)x^2}{1+(n-1)x}\right)$$

$$= \frac{nx^2}{1+nx}.$$

$$\therefore f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx^2}{1+nx} = \lim_{n\to\infty} \frac{x^2}{\frac{1}{n}+x}$$

$$= \begin{cases} x & \text{if } 0 < x \le 1 \\ 0 & \text{if } x = 0 \end{cases}$$

$\therefore$ $f_n \to f$ pointwise on $[0,1]$.

For uniform convergence, let $\epsilon > 0$ be given then for $0 < x \le 1$, we have

$$|f_n(x) - f(x)| = \left|\frac{nx^2}{1+nx} - x\right| = \left|\frac{-x}{1+nx}\right| = \frac{x}{1+nx} < \epsilon$$

if $1+nx > \frac{x}{\epsilon}$

if $n > \frac{1}{\epsilon} - \frac{1}{x}$

Now max value of $\frac{1}{\epsilon} - \frac{1}{x}$ on $(0,1]$ is $\frac{1}{\epsilon} - 1$

$\therefore$ We can choose a the integer m s.t

$$m > \frac{1}{\epsilon} - 1 \ge \frac{1}{\epsilon} - \frac{1}{x} \quad \forall x \in (0,1]$$

then we have $|f_n(x) - f(x)| < \epsilon \quad \forall n \ge m$
$\forall x \in (0,1]$

Also for $x = 0$, $|f_n(x) - f(x)| = |0 - 0| = 0 < \epsilon \quad \forall n \ge m$.

Thus $|f_n(x) - f(x)| < \epsilon \quad \forall\, n \geq m, \ \forall\, x \in [0,1]$

Hence seq $\{f_n\}$ converges uniformly on $[0,1]$

$\Rightarrow$ series ΣU_n converges uniformly on $[0,1]$.

Q. Show that $x=0$ is point of non uniform convergence of series

$$x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \cdots$$

Sol^n Let given series is ΣU_n and $f_n = U_1 + U_2 + \cdots + U_n$ be nth partial sum of ΣU_n.

$\therefore f_n(x) = x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \cdots$ n terms

which forms a G.P.

$$\therefore f_n(x) = \frac{x^2\left[1 - \frac{1}{(1+x^2)^n}\right]}{1 - \frac{1}{1+x^2}} = (1+x^2)\left[1 - \left(\frac{1}{1+x^2}\right)^n\right]$$

$$\therefore f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 1+x^2 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$$\left[\begin{array}{l} \because 1+x^2 \geq 1 \Rightarrow 0 < \frac{1}{1+x^2} \leq 1 \\ \Rightarrow \left(\frac{1}{1+x^2}\right)^n \to 0 \text{ as } n \to \infty \end{array}\right]$$

for uniform convergence

let $\epsilon > 0$ be given, then for $x \neq 0$, we have

$$|f_n(x) - f(x)| = \left|(1+x^2) - \frac{1}{(1+x^2)^{n-1}} - (1+x^2)\right| = \frac{1}{(1+x^2)^{n-1}} < \epsilon$$

if $(1+x^2)^{n-1} > \frac{1}{\epsilon}$ if $(n-1)\log(1+x^2) > \log\frac{1}{\epsilon}$

if $n-1 > \frac{\log\frac{1}{\epsilon}}{\log(1+x^2)}$ if $n > 1 + \frac{\log\frac{1}{\epsilon}}{\log(1+x^2)}$

Since $1 + \frac{\log\frac{1}{\epsilon}}{\log(1+x^2)} \longrightarrow \infty$ as $x \to 0$

$\therefore$ We can not choose a +ve integer m s.t

$$|f_n(x) - f(x)| < \epsilon \quad \forall\, n \geq m$$
$$\forall\, x \in [0,1]$$

$\therefore \{f_n\}$ is not ~~ut~~ uniformly cgt on any interval containing 0

$\Rightarrow$ $x=0$ is pt of non uniform convergence of $\{f_n\}$ and hence of the given series.

Q. Show that the series

$$x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \cdots$$ is not uniformly convergent on $[0,1]$.

Sol^n: let given series is ΣU_n and $f_n = U_1 + U_2 + \cdots + U_n$ be nth partial sum of ΣU_n.

$$\therefore \quad f_n(x) = x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \cdots + \frac{x^4}{(1+x^4)^{n-1}}$$

which is G.P series

$$= \frac{x^4\left[1 - \frac{1}{(1+x^4)^n}\right]}{1 - \frac{1}{1+x^4}} = (1+x^4)\left[1 - \frac{1}{(1+x^4)^n}\right]$$

$$\therefore \quad f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 1+x^4 & \text{when } 0 < x \leq 1 \\ 0, & \text{when } x = 0 \end{cases}$$

$\therefore f_n \to f$ point wise on $[0,1]$.

for uniform convergence, let $\epsilon > 0$ be given and for $0 < x \leq 1$, we have

$$|f_n(x) - f(x)| = \left|(1+x^4) - \frac{1}{(1+x^4)^{n-1}} - (1+x^4)\right| = \frac{1}{(1+x^4)^{n-1}} < \epsilon$$

if $(1+x^4)^{n-1} > \frac{1}{\epsilon}$ if $(n-1)\log(1+x^4) > \log\frac{1}{\epsilon}$

if $n > 1 + \dfrac{\log\frac{1}{\epsilon}}{\log(1+x^4)}$

When $x \to 0$, then $1 + \dfrac{\log \frac{1}{\epsilon}}{\log (1+x^4)} \to \infty$

$\therefore$ We can not choose a +ve integer m s.t

$|f_n(x) - f(x)| < \epsilon \quad \forall\, n \geq m, \ \forall\, x \in [0,1]$.

$\therefore$ Seq $\{f_n\}$ is not uniformly cgt on $[0,1]$

$\Rightarrow \sum U_n$ is not uniformly cgt on $[0,1]$.

Q. Show that series

$$\frac{x}{1+x} + \frac{x}{(1+x)(1+2x)} + \frac{x}{(1+2x)(1+3x)} + \cdots$$ is uniformly cgt on $[a,\infty)$, $a>0$. Show also that series is non uniformly cgt near $x=0$.

Sol^n: Let given series is $\sum U_n$ and $f_n = U_1 + U_2 + \cdots + U_n$ be nth partial sum of $\sum U_n$.

$$\therefore f_n(x) = \frac{x}{1+x} + \frac{x}{(1+x)(1+2x)} + \cdots + \frac{x}{(1+\overline{n-1}\,x)(1+nx)}$$

$$= \left(1 - \frac{1}{1+x}\right) + \left(\frac{1}{x+1} - \frac{1}{2x+1}\right) + \left(\frac{1}{2x+1} - \frac{1}{3x+1}\right) + \cdots + \left(\frac{1}{(n-1)x+1} - \frac{1}{nx+1}\right)$$

$$= 1 - \frac{1}{nx+1} = \frac{nx}{nx+1}$$

$$\therefore f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{nx+1} = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$\therefore f_n \to f$ point wise on R or $[0,\infty)$

For uniform convergence, let $\epsilon > 0$ be given then for $x > 0$ we have

$$|f_n(x) - f(x)| = \left|\frac{nx}{nx+1} - 1\right| = \frac{1}{nx+1} < \epsilon$$

if $nx+1 > \frac{1}{\epsilon}$ if $n > \frac{1}{x}\left(\frac{1}{\epsilon} - 1\right)$

Now $\frac{1}{x}\left(\frac{1}{\epsilon} - 1\right) \to \infty$, when $x \to 0$

$\therefore$ It is not possible to choose a +ve integer m

S.t $|f_n(x) - f(x)| < \epsilon \quad \forall n \geq m$
$\forall x \in [0, \infty)$

$\therefore$ $\{f_n\}$ is not uniformly cgt near $x=0$

$\Rightarrow$ ΣU_n is not uniformly cgt near $x=0$

Also Max value of $\frac{1}{x}\left(\frac{1}{\epsilon}-1\right)$ on $[a,\infty)$ is $\frac{1}{a}\left(\frac{1}{\epsilon}-1\right)$

we can choose a +ve integer m. S.t

$$m > \frac{1}{a}\left(\frac{1}{\epsilon}-1\right) \geq \frac{1}{x}\left(\frac{1}{\epsilon}-1\right) \quad \forall x \in [a,\infty)$$

then $|f_n(x) - f(x)| < \epsilon \quad \forall n \geq m$ and $\forall x \in [a,\infty)$

$\therefore$ $f_n \to f$ uniformly on $[a,\infty)$

$\Rightarrow$ ΣU_n is uniformly cgt on $[a,\infty)$

Q. Show that $x=0$ is a pt of non uniform convergence of the series $\sum_{n=1}^{\infty} \frac{-2x(1+x)^{n-1}}{[1+(1+x)^{n-1}][1+(1+x)^n]}$.

Sol^n: let given series is ΣU_n and $f_n = U_1 + U_2 + \cdots + U_n$ be nth partial sum of ΣU_n.

$$\therefore \quad U_n(x) = \frac{-2x(1+x)^{n-1}}{[1+(1+x)^{n-1}][1+(1+x)^n]} = \frac{2}{1+(1+x)^n} - \frac{2}{1+(1+x)^{n-1}} \quad \text{---(1)}$$

Putting $n = 1,2,3,\ldots n$ in (1), we get

$$U_1(x) = \frac{2}{1+(1+x)} - 1$$

$$U_2(x) = \frac{2}{1+(1+x)^2} - \frac{2}{1+(1+x)}$$

$$U_3(x) = \frac{2}{1+(1+x)^3} - \frac{2}{1+(1+x)^2}$$

$$U_n(x) = \frac{2}{1+(1+x)^n} - \frac{2}{1+(1+x)^{n-1}}$$

Adding, we get

$$f_n(x) = \frac{2}{1+(1+x)^n} - 1$$

$$\therefore f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} -1 & \text{if } x>0 \\ 0 & \text{if } x=0 \\ 1 & \text{if } x<0 \end{cases}$$

Let $\epsilon>0$ be given, then for $x>0$, we have

$$|f_n(x) - f(x)| = \left|\frac{2}{1+(1+x)^n} - 1 + 1\right| = \frac{2}{1+(1+x)^n} < \epsilon$$

if $1+(1+x)^n > \frac{2}{\epsilon}$ if $(1+x)^n > \frac{2}{\epsilon} - 1$

if $n \log(1+x) > \log\left(\frac{2}{\epsilon}-1\right)$

if $n > \dfrac{\log\left(\frac{2}{\epsilon}-1\right)}{\log(1+x)}$

~~Now~~ when $x\to 0$, then $\dfrac{\log\left(\frac{2}{\epsilon}-1\right)}{\log(1+x)} \to \infty$

$\therefore$ It is not possible to choose a +ve integer m s.t

$$|f_n(x) - f(x)| < \epsilon \quad \forall\, n \geq m,\ \forall\, x \in [0,\infty)$$

$\Rightarrow$ $\{f_n\}$ is not uniformly cgt on any interval containing '0'.

$\Rightarrow$ 0 is pt. of non uniform convergence for $\{f_n\}$ hence for Σu_n

Th^m M_n-Test for uniform convergence of seq of functions.

Statement: Let $\{f_n\}$ be seq of functions defined on E such that $\lim_{n\to\infty} f_n(x) = f(x)$ $\forall x \in E$ i.e. $f_n \to f$ pointwise on E and $M_n = \sup_{x\in E} |f_n(x) - f(x)|$ $\forall n \in N$.

Then $f_n \to f$ uniformly on E iff the seq $\{M_n\}$ of real no is null seq i.e. $M_n \to 0$ as $n \to \infty$.

Proof: Given $\lim_{n\to\infty} f_n(x) = f(x)$ $\forall x \in E$

and $M_n = \sup_{x\in E} |f_n(x) - f(x)|$ $\forall n \in N$ ——— (1)

First let $f_n \to f$ uniformly on E

$\therefore$ given $\epsilon > 0$ $\exists$ $m \in N$ s.t

$|f_n(x) - f(x)| < \epsilon$ $\forall n \geq m$ and $\forall x \in E$

$\Rightarrow$ $\sup_{x\in E} |f_n(x) - f(x)| < \epsilon$ $\forall n \geq m$

$\Rightarrow$ $M_n < \epsilon$ $\forall n \geq m$ (by (1))

$\Rightarrow$ $|M_n - 0| < \epsilon$ $\forall n \geq m$ $[\because M_n \geq 0 \ \forall n \in N]$

$\Rightarrow$ $\{M_n\}$ is null seq of real no.

Conversely let $\{M_n\}$ be null seq of real no.

i.e. $M_n \to 0$ as $n \to \infty$

$\therefore$ given $\epsilon > 0$ $\exists$ $m \in N$ s.t

$|M_n - 0| < \epsilon$ $\forall n \geq m$.

$\Rightarrow$ $M_n < \epsilon$ $\forall n \geq m$ $[\because M_n \geq 0 \ \forall n \in N]$

$\Rightarrow$ $\sup_{x\in E} |f_n(x) - f(x)| < \epsilon$ $\forall n \geq m$ (by (1))

$\Rightarrow$ $|f_n(x) - f(x)| < \epsilon$ $\forall n \geq m$ and $\forall x \in E$

$\Rightarrow$ $f_n \to f$ uniformly on E.

Th^m: Weierstrass's M-Test for uniform Convergence of Series of function:

Statement: Let Σf_n be series of functions defined on E s.t $|f_n(x)| \le M_n$ $\forall x \in E$ and $\forall n \in N$.

Then Σf_n Converges uniformly on E if ΣM_n of +ve real no is cgt.

Proof: → Given that Σf_n be series of functions defined on E s.t

$$|f_n(x)| \le M_n \quad \forall x \in E \text{ and } \forall n \in N \quad \text{—(1)}$$

Let ΣM_n of real no is cgt

∴ by Cauchy's general principal for convergence of series of real no.

given $\epsilon > 0$ $\exists$ $m \in N$ s.t

$$|M_{n+1} + M_{n+2} + \cdots + M_{n+p}| < \epsilon \quad \forall n \ge m, p \ge 1$$

$$\Rightarrow M_{n+1} + M_{n+2} + \cdots + M_{n+p} < \epsilon \quad \forall n \ge m, p \ge 1 \quad \text{—(2)}$$

$[\because M_n \ge 0 \ \forall n \in N]$

Now for $x \in E$ and $\forall$ $n \ge m$, $p \ge 1$ we have

$$|f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+p}(x)|$$

$$\le |f_{n+1}(x)| + |f_{n+2}(x)| + \cdots + |f_{n+p}(x)|$$

$$\le M_{n+1} + M_{n+2} + \cdots + M_{n+p} \quad \forall x \in E \quad \text{(by (1))}$$

$$< \epsilon \quad \forall n \ge m, p \ge 1 \text{ and } \forall x \in E \quad \text{(by (2))}$$

∴ By Cauchy's criterion for uniform convergence of series of function

Σf_n is uniformly cgt on E.

Q. Show that seq of functions $\{f_n\}$, where $f_n(x) = nx(1-x)^n$ is not uniformly cgt on $[0,1]$.

Sol^n: Here $f_n(x) = nx(1-x)^n$, $x \in [0,1]$.

For $0<x<1$.

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{(1-x)^{-n}} \quad \left(\tfrac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{n\to\infty} \frac{x}{-(1-x)^{-n}\log(1-x)}$$

$$= \lim_{n\to\infty} \frac{-x(1-x)^n}{\log(1-x)}$$

$$= 0 \qquad [\because (1-x)^n \to 0 \text{ as } n\to\infty]$$

Also when $x=0$ and 1, then $f_n(x) = 0 \ \forall n \in N$.

$\therefore f(x) = 0 \ \forall x \in [0,1]$.

Now $|f_n(x) - f(x)| = |nx(1-x)^n - 0| = nx(1-x)^n$

Let $y = nx(1-x)^n$

$$\therefore \frac{dy}{dx} = n(1-x)^n - n^2x(1-x)^{n-1}$$

$$= n(1-x)^{n-1}[(1-x) - nx] = n(1-x)^{n-1}[1-(n+1)x]$$

For Max. or Min $\frac{dy}{dx} = 0 \Rightarrow n(1-x)^{n-1}[1-(n+1)x] = 0$

$$\Rightarrow x = \frac{1}{n+1}$$

Also $$\frac{d^2y}{dx^2} = -n(n-1)(1-x)^{n-2}[1-(n+1)x] - n(n+1)(1-x)^{n-1}$$

$$\therefore \left.\frac{d^2y}{dx^2}\right]_{x=\frac{1}{n+1}} = -n(n-1)\left(1-\frac{1}{n+1}\right)^{n-2}\left[1-\frac{n+1}{n+1}\right] - n(n+1)\left(1-\frac{1}{n+1}\right)^{n-1}$$

$$= -n(n+1)\left(\frac{n}{n+1}\right)^{n-1} < 0$$

Which shows that y is max at $x = \frac{1}{n+1}$ and

$$y_{max} = \frac{n}{n+1}\left(1-\frac{1}{n+1}\right)^n = \frac{n}{n+1}\left(\frac{n}{n+1}\right)^n$$

$$= \left(\frac{n}{n+1}\right)^{n+1} = \left(1-\frac{1}{n+1}\right)^{n+1}$$

Also $x = \frac{1}{n+1} \to 0$ as $n \to \infty$

$\therefore M_n = \sup_{x \in [0,1]} |f_n(x) - f(x)|$

$$= \left(1-\frac{1}{n+1}\right)^{n+1} \to \frac{1}{e} \quad \text{as } n \to \infty$$

Since $M_n \not\to 0$ as $n \to \infty$

By M_n-Test, $\{f_n\}$ is not uniformly cgt on $[0,1]$.

Q. Show that 0 is a pt of non uniform convergence for seq. $\{f_n\}$ where $f_n(x) = nx\,e^{-nx^2}$, $x \in R$.

Sol^n: Here $f_n(x) = nx\,e^{-nx^2}$, $x \in R$

For $x \in R - \{0\}$

$\therefore f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{e^{nx^2}}$ $\left(\frac{\infty}{\infty}\text{ form}\right)$

$$= \lim_{n\to\infty} \frac{x}{x^2 e^{nx^2}} = 0 \quad \forall\, x \in R - \{0\}$$

At $x = 0$, $f_n(x) = 0 \quad \forall\, n \in N$.

$\therefore f(x) = 0 \quad \forall\, x \in R$.

Now $|f_n(x) - f(x)| = |nx\,e^{-nx^2} - 0|$

$$= nx\,e^{-n^2x^2} \quad \forall\, x \geq 0.$$

Let $y = nx\,e^{-nx^2}$

$\therefore \frac{dy}{dx} = ne^{-nx^2} + nx\,e^{-nx^2}(-2nx)$

$$= ne^{-nx^2}(1-2nx^2)$$

For Max. or Min $\frac{dy}{dx} = 0 \Rightarrow ne^{-nx^2}(1-2nx^2) = 0$

$\Rightarrow 1 - 2nx^2 = 0 \Rightarrow x = \frac{1}{\sqrt{2n}}$

Also $\frac{d^2y}{dx^2} = n \cdot e^{-nx^2}(-2nx) - 2n^2(2x)\cdot e^{-nx^2}$
$$- 2n^2x^2 \cdot e^{-nx^2}(-2nx)$$
$$= -2n^2x\, e^{-nx^2} - 4n^2x \cdot e^{-nx^2} + 4n^3x^3 \cdot e^{-nx^2}$$
$$= -2n^2x\, e^{-nx^2}(3 - 2nx^2)$$

$$\therefore \left.\frac{d^2y}{dx^2}\right]_{x=\frac{1}{\sqrt{2n}}} = -\frac{2n^2}{\sqrt{2n}}\, e^{-\frac{n}{2n}}\left(3 - \frac{2n}{2n}\right)$$
$$= -\frac{4n^2}{\sqrt{2n}}\, e^{-1/2} < 0.$$

Which shows that y is max. when $x = \frac{1}{\sqrt{2n}}$.

and $y_{max.} = n \cdot \frac{1}{\sqrt{2n}}\, e^{-\frac{n}{2n}} = \left(\frac{n}{2e}\right)^{1/2}$

Also $x = \frac{1}{\sqrt{2n}} \to 0$ as $n \to \infty$

$\therefore M_n = \sup\limits_{x \geq 0} |f_n(x) - f(x)| = \left(\frac{n}{2e}\right)^{\frac{1}{2}}$

As $M_n \to \infty$ as $n \to \infty$

$\therefore M_n \not\to 0$ as $n \to \infty$ $\therefore$ by M_n-Test

seq $\{f_n\}$ is not uniformly cgt on $[0,k], k>0$

$\therefore$ 0 is pt of non uniform convergence for $\{f_n\}$

Q. Show that seq $\{f_n\}$ where $f_n(x) = \frac{\sin nx}{\sqrt{n}}$ is uniformly cgt on $[0, 2\pi]$.

Solⁿ Here $f_n(x) = \frac{\sin nx}{\sqrt{n}}$

$\therefore f(x) = \lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \frac{\sin nx}{\sqrt{n}}$ $\qquad |\sin nx| \leq 1 \;\forall x \in [0, 2\pi]$

$$= 0 \quad \forall x \in [0, 2\pi].$$

$\therefore M_n = \sup\{|f_n(x) - f(x)| : x \in [0, 2\pi]\}$
$$= \sup\left\{\left|\frac{\sin nx}{\sqrt{n}}\right| : x \in [0, 2\pi]\right\}$$

$\therefore \; M_n = \frac{1}{\sqrt{n}}$ $\left[\because \text{Max value of } \sin nx \text{ is } 1 \text{ when } x = \frac{\pi}{2n}\right]$

$\therefore \; M_n \to 0$ as $n \to \infty$

$\therefore$ By M_n - Test, $\{f_n\}$ is uniformly cgt on $[0, 2\pi]$

Q. Show that seq $\{f_n\}$ where $f_n(x) = \frac{x}{n(1+nx^2)}$ is uniformly cgt on R.

Sol^n: Here $f_n(x) = \frac{x}{n(1+nx^2)}$, $x \in R$.

$$\therefore \; f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{n(1+nx^2)} = 0 \quad \forall x \in R$$

$\therefore$ $f_n \to f$ point wise on R.

Now $$|f_n(x) - f(x)| = \left|\frac{x}{n(1+nx^2)} - 0\right| = \left|\frac{x}{n(1+nx^2)}\right|$$

Let $y = \frac{x}{n(1+nx^2)}$ $\therefore$ $$\frac{dy}{dx} = \frac{n(1+nx^2) - x(2n^2x)}{[n(1+nx^2)]^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{n + n^2x^2 - 2n^2x^2}{[n(1+nx^2)]^2} = \frac{n - n^2x^2}{[n(1+nx^2)]^2}$$

for Max. or Min. $\frac{dy}{dx} = 0 \Rightarrow \frac{n-n^2x^2}{[n(1+nx^2)]^2} = 0$

$$\Rightarrow n - n^2x^2 = 0 \Rightarrow x = \frac{1}{\sqrt{n}}$$

Also $$\frac{d^2y}{dx^2} = \frac{[n(1+nx^2)]^2(-2n^2x) - (n-n^2x^2)\,2n(1+nx^2)(2n^2x)}{[n(1+nx^2)]^4}$$

$$= \frac{-2n^2x(1+nx^2)\left[n^2(1+nx^2) + 2n(n-n^2x^2)\right]}{[n(1+nx^2)]^4}$$

$$= \frac{-2nx\left[n^2 + n^3x^2 + 2n^2 - 2n^3x^2\right]}{[n(1+nx^2)]^3}$$

$$= \frac{-2nx(3n^2 - n^3x^2)}{[n(1+nx^2)]^3} = \frac{-2n^3x(3-nx^2)}{[n(1+nx^2)]^3}$$

$$\therefore \left.\frac{d^2y}{dx^2}\right]_{x=\frac{1}{\sqrt{n}}} = \frac{-2n^3\cdot\frac{1}{\sqrt{n}}\left(3-n\cdot\frac{1}{n}\right)}{\left[n\left(1+n\cdot\frac{1}{n}\right)\right]^3} = \frac{-2n^{5/2}(2)}{8n^3}$$

$$= -\frac{1}{2n^{1/2}} < 0$$

Which shows that y is max. when $x = \frac{1}{\sqrt{n}}$

and $y_{max} = \dfrac{\frac{1}{\sqrt{n}}}{n\left(1+n\cdot\frac{1}{n}\right)} = \dfrac{1}{2n^{3/2}}$

Also $x = \frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$

$\therefore M_n = \sup\limits_{x\in R} |f_n(x) - f(x)| = \dfrac{1}{2n^{3/2}}$

$\because M_n \to 0$ as $n \to \infty$

$\therefore$ By M_n-Test, seq $\{f_n\}$ is uniformly cgt on R.

Q. Show that seq of function $\{f_n\}$, where $f_n(x) = \dfrac{x}{(n+x^2)^2}$ is uniformly cgt for $x \geq 0$.

Soln: Here $f_n(x) = \dfrac{x}{(n+x^2)^2}$, $x \geq 0$.

$\therefore f(x) = \lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \dfrac{x}{(n+x^2)^2} = 0 \quad \forall x \geq 0.$

$\therefore f_n \to f$ point wise for $x \geq 0$.

Now $|f_n(x) - f(x)| = \left|\dfrac{x}{(n+x^2)^2} - 0\right| = \dfrac{x}{(n+x^2)^2}$

Let $y = \dfrac{x}{(n+x^2)^2}$ $\therefore \dfrac{dy}{dx} = \dfrac{(n+x^2)^2 - x[2(n+x^2)(2x)]}{(n+x^2)^4}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{(n+x^2)[n+x^2-4x^2]}{(n+x^2)^4} = \dfrac{n-3x^2}{(n+x^2)^3}$

For Max or Min $\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{n-3x^2}{(n+x^2)^3} = 0$

$\Rightarrow \quad n - 3x^2 = 0 \quad \Rightarrow \quad x = \sqrt{\frac{n}{3}}$

Also $\dfrac{d^2y}{dx^2} = \dfrac{(n+x^2)(-6x) - (n-3x^2)(2x)}{(n+x^2)^2}$

$$= \frac{-6x(n+x^2) - 2x(n-3x^2)}{(n+x^2)^2} = \frac{-8nx}{(n+x^2)^2}$$

$$\therefore \left.\frac{d^2y}{dx^2}\right]_{x=\sqrt{\frac{n}{3}}} = \frac{-8n\sqrt{\frac{n}{3}}}{\left(n+\frac{n}{3}\right)^2} < 0$$

which show that y is max. when $x = \sqrt{\frac{n}{3}}$

and $y_{max} = \dfrac{\sqrt{\frac{n}{3}}}{\left(n+\frac{n}{3}\right)^2} = \dfrac{\sqrt{n}}{\sqrt{3}} \times \dfrac{9}{16n^2} = \dfrac{3\sqrt{3}}{16n^{3/2}}$

$\therefore M_n = \sup\limits_{x \geq 0} |f_n(x) - f(x)| = \dfrac{3\sqrt{3}}{16n^{3/2}} \to 0$ as $n \to \infty$

By M_n-Test seq $\{f_n\}$ is uniformly cgt

$\forall x \geq 0$

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(8): Show that seq. $\{f_n\}$, where $f_n(x) = \dfrac{nx}{1+n^2x^2}$ is not uniformly cgt on any interval containing '0'.

Solⁿ: Here $f_n(x) = \dfrac{nx}{1+n^2x^2}$

$\therefore f(x) = \lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \dfrac{nx}{1+n^2x^2} = \lim\limits_{n\to\infty} \dfrac{\frac{x}{n}}{\frac{1}{n^2}+x^2}$

$= 0 \quad \forall x \in R.$

Now $|f_n(x) - f(x)| = \left|\dfrac{nx}{1+n^2x^2} - 0\right| = \left|\dfrac{nx}{1+n^2x^2}\right|$

Let $y = \dfrac{nx}{1+n^2x^2} \quad \therefore \dfrac{dy}{dx} = \dfrac{(1+n^2x^2)(n) - nx(2n^2x)}{(1+n^2x^2)^2}$

$$= \frac{n - n^3x^2}{(1+n^2x^2)^2}$$

for Max. or Min $\frac{dy}{dx} = 0 \Rightarrow \frac{n - n^3x^2}{(1+n^2x^2)^2} = 0$

$\Rightarrow x = \frac{1}{n}$

Also $\frac{d^2y}{dx^2} = \frac{(1+n^2x^2)^2(-2n^3x) - (n - n^3x^2)\,2(1+n^2x^2)\cdot 2n^2x}{(1+n^2x^2)^4}$

$= \frac{-2n^2x(1+n^2x^2)\left[n(1+n^2x^2) - 2(n - n^3x^2)\right]}{(1+n^2x^2)^4}$

$= \frac{-2n^2x(3n^3x^2 - n)}{(1+n^2x^2)^3}$

$\therefore \left.\frac{d^2y}{dx^2}\right]_{x=\frac{1}{n}} = \frac{-2n(3n-n)}{(1+1)^3} = \frac{-n^2}{2} < 0$

which shows that y is max when $x = \frac{1}{n}$ and

$y_{max} = \frac{n\cdot\frac{1}{n}}{1+n^2\cdot\frac{1}{n^2}} = \frac{1}{2}$

Also $x = \frac{1}{n} \to 0$ as $n \to \infty$

Let us take an interval $[a,b]$ containing 0

$\therefore M_n = \sup_{x\in[a,b]} |f_n(x) - f(x)| = \frac{1}{2}$

$\therefore M_n$ does not tend to zero as $n \to \infty$

By M_n Test, seq $\{f_n\}$ is not uniformly cgt in any interval containing '0'.

Q. Test the seq $\{f_n\}$ where $f_n(x) = \frac{nx}{1+n^3x^2}$ for uniform convergence over $[0,1]$.

Sol^n Here $f_n(x) = \frac{nx}{1+n^3x^2}$, $x \in [0,1]$

$\therefore f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+n^3x^2} = \lim_{n\to\infty} \frac{\frac{x}{n^2}}{\frac{1}{n^3}+x^2} = 0 \quad \forall x\in R$

Now $\left|f_n(x) - f(x)\right| = \left|\frac{nx}{1+n^3x^2} - 0\right| = \left|\frac{nx}{1+n^3x^2}\right|$

Let $y = \frac{nx}{1+n^3x^2}$ $\therefore \frac{dy}{dx} = \frac{(1+n^3x^2)n - nx(2n^3x)}{(1+n^3x^2)^2}$

$\Rightarrow \frac{dy}{dx} = \frac{n - n^4x^2}{(1+n^3x^2)^2}$

for Max. or Min, $\frac{dy}{dx} = 0 \Rightarrow \frac{n-n^4x^2}{(1+n^3x^2)^2} = 0 \Rightarrow x = \frac{1}{n^{3/2}}$

Also $\frac{d^2y}{dx^2} = \frac{(1+n^3x^2)^2(-2n^4x) - (n-n^4x^2)\cdot 2(1+n^3x^2)(2n^3x)}{(1+n^3x^2)^4}$

$$= \frac{-2n^3x(1+n^3x^2)\left[n(1+n^3x^2) + 2(n-n^4x^2)\right]}{(1+n^3x^2)^4}$$

$$= \frac{-2n^3x(3n - n^4x^2)}{(1+n^3x^2)^3}$$

$\therefore \left.\frac{d^2y}{dx^2}\right]_{x=\frac{1}{n^{3/2}}} = \frac{-2n^3\left(\frac{1}{n^{3/2}}\right)\left[3n - \frac{n^4}{n^3}\right]}{\left(1+\frac{n^3}{n^3}\right)^3} =$

$$= -\frac{n^{5/2}}{2} < 0$$

which shows that y is max when $x = \frac{1}{n^{3/2}}$

and $y_{max} = \frac{n\cdot\frac{1}{n^{3/2}}}{1+n^3\cdot\frac{1}{n^3}} = \frac{1}{2\sqrt{n}}$

Also $x = \frac{1}{n^{3/2}} \to 0$ as $n \to \infty$

$\therefore M_n = \sup\limits_{x\in[0,1]} \left|f_n(x) - f(x)\right| = \frac{1}{2\sqrt{n}} \to 0$ as $n \to \infty$

$\therefore$ by M_n-Test, $\{f_n\}$ converges uniformly to f on $[0,1]$.

—o—

Q.: Show that seq $\{f_n\}$, where $f_n(x) = \dfrac{n^2 x}{1+n^4x^2}$ does not converge uniformly on $[0,1]$

Sol^n: Here $f_n(x) = \dfrac{n^2 x}{1+n^4x^2}$, $x \in [0,1]$.

$$\therefore f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{n^2 x}{1+n^4x^2} = \lim_{n\to\infty} \frac{\frac{x}{n^2}}{\frac{1}{n^4}+x^2}$$

$$= 0 \quad \forall x \in [0,1].$$

Now $|f_n(x) - f(x)| = \left|\dfrac{n^2x}{1+n^4x^2} - 0\right| = \dfrac{n^2 x}{1+n^4x^2}$.

Let $y = \dfrac{n^2x}{1+n^4x^2}$ $\therefore \dfrac{dy}{dx} = \dfrac{n^2(1-n^4x^2)}{(1+n^4x^2)^2}$ (After solving)

for Max or Min $\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{n^2(1-n^4x^2)}{(1+n^4x^2)^2} = 0$

$$\Rightarrow x = \frac{1}{n^2}$$

Also $\dfrac{d^2y}{dx^2} = \dfrac{-2n^6x(3-n^4x^2)}{(1+n^4x^2)^3}$ (After solving)

$\therefore \left.\dfrac{d^2y}{dx^2}\right]_{x=\frac{1}{n^2}} = -\dfrac{n^4}{2} < 0$ (After solving)

which shows that y is max. when $x = \frac{1}{n^2}$

and $y_{max} = \dfrac{n^2 \cdot \frac{1}{n^2}}{1+n^4 \cdot \frac{1}{n^4}} = \dfrac{1}{2}$

Also $x = \frac{1}{n^2} \to 0$ as $n \to \infty$

$\therefore M_n = \sup_{x\in[0,1]} |f_n(x) - f(x)| = \frac{1}{2}$.

$\therefore M_n \not\to 0$ as $n \to \infty$

By M_n-Test, seq $\{f_n\}$ does not converge uniformly on $[0,1]$

——o——

Q. Show that seq $\{f_n\}$ where $f_n(x) = \frac{x^n}{1+x^n}$ converges uniformly on $[0,\frac{1}{2}]$.

Sol^n: Here $f_n(x) = \frac{x^n}{1+x^n}$, $x \in [0,\frac{1}{2}]$

$\therefore f(x) = \lim_{n\to\infty} \frac{x^n}{1+x^n} = 0$ $\left[\because \lim_{n\to\infty} x^n = 0 \;\; \forall x \in [0,\frac{1}{2}]\right]$

$\therefore M_n = \sup_{x\in[0,\frac{1}{2}]} |f_n(x) - f(x)|$

$= \sup_{x\in[0,\frac{1}{2}]} \left|\frac{x^n}{1+x^n}\right|$

$\Rightarrow M_n \le \frac{1}{2^n} \quad \forall x \in [0,\frac{1}{2}]$ $\left[\because \frac{x^n}{1+x^n} \le x^n \le \frac{1}{2^n} \;\; \forall x \in [0,\frac{1}{2}]\right]$

Since $\frac{1}{2^n} \to 0$ as $n \to \infty$

$\therefore M_n \to 0$ as $n \to \infty$

$\therefore$ By M_n-Test, $\{f_n\}$ converges uniformly on $[0,\frac{1}{2}]$.

Q. Prove that the seq $\{f_n\}$ where $f_n(x) = x^{n-1}(1-x)$ converges uniformly on $[0,1]$.

Sol^n: $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^{n-1}(1-x) = 0 \quad \forall x \in [0,1]$

$\left[\lim_{n\to\infty} x^n = 0,\; x<1\right]$

Now $|f_n(x) - f(x)| = |x^{n-1}(1-x) - 0| = |x^{n-1}(1-x)| = x^{n-1}(1-x)$

Let $y = x^{n-1}(1-x)$

$\therefore \frac{dy}{dx} = (n-1)x^{n-2}(1-x) - x^{n-1} = x^{n-2}[(n-1) - nx]$

For Max or Min. $\frac{dy}{dx} = 0 \Rightarrow x = 0$, or $\frac{n-1}{n}$

Also $\frac{d^2y}{dx^2} = (n-2)x^{n-3}[(n-1)-nx] - nx^{n-2}$

$$\therefore \left.\frac{d^2y}{dx^2}\right]_{x=\frac{n-1}{n}} = (n-2)\left(\frac{n-1}{n}\right)^{n-3}\left[(n-1) - \frac{n(n-1)}{n}\right] - n\left(\frac{n-1}{n}\right)^{n-2}$$

$$= -n\left(\frac{n-1}{n}\right)^{n-2} < 0$$

Which shows that y is max. when $x = \frac{n-1}{n}$

and $y_{max} = \left(\frac{n-1}{n}\right)^{n-1}\left(1 - \frac{n-1}{n}\right) = \frac{1}{n}\left(1-\frac{1}{n}\right)^{n-1}$

$$\therefore M_n = \sup_{x \in [0,1]} |f_n(x) - f(x)| = \frac{1}{n}\left(1-\frac{1}{n}\right)^{n-1}$$

$$= \frac{1}{n}\left(1-\frac{1}{n}\right)^{n}\left(1-\frac{1}{n}\right)^{-1} \to 0 \times \frac{1}{e} \times 1 = 0 \quad \text{as } n \to \infty$$

By M_n-Test, seq. $\{f_n\}$ is uniformly cgt on $[0,1]$.

Q.

Q. Show by M_n-test '0' is pt. of non uniform convergence of seq $\{f_n\}$ where $f_n(x) = 1-(1-x^2)^n$.

Sol^n

$$f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } x=0 \\ 1 & \text{if } 0<|x|<\sqrt{2} \end{cases}$$

for $x \in (0,\sqrt{2})$

$$|f_n(x)-f(x)| = |1-(1-x^2)^n - 1| = |(1-x^2)^n|$$

Let $y = (1-x^2)^n \quad \therefore \quad \dfrac{dy}{dx} = n(1-x^2)^{n-1}(-2x)$

$$\frac{d^2y}{dx^2} = -2n\left[(1-x^2)^{n-1} + (n-1)(1-x^2)^{n-2}(-2x)\,x\right]$$

$$= -2n(1-x^2)^{n-2}\left[1-x^2+2x^2-2nx^2\right]$$

$$= -2n(1-x^2)^{n-2}\left[1+x^2-2nx^2\right]$$

for Max or Min $\dfrac{dy}{dx} = 0 \Rightarrow -2nx(1-x^2)^{n-1} = 0$

$$\Rightarrow x = 0$$

$$\left.\frac{d^2y}{dx^2}\right]_{x=0} = -2n < 0$$

$\therefore$ y is max. at $x=0$ and $y_{max.} = 1$

$\therefore$ $M_n = 1 \not\to 0$ as $n\to\infty$

$\therefore$ By M_n-Test seq $\{f_n\}$ is not uniformly cgt on any interval containing '0'.

$\therefore$ '0' is pt of non uniform convergence for seq $\{f_n\}$

Q. Show that the series

$$\frac{x}{1+x^2} + \left(\frac{2^2x}{1+2^3x^2} - \frac{x}{1+x^2}\right) + \left(\frac{3^2x}{1+3^3x^2} - \frac{2^2x}{1+2^3x^2}\right) + \cdots$$

does not converge uniformly on $[0,1]$.

Sol^n: Let given series is ΣU_n and $f_n = U_1 + U_2 + \cdots + U_n$ be nth partial sum of ΣU_n.

$$\therefore\ U_1(x) = \frac{x}{1+x^2}$$

$$U_2(x) = \frac{2^2x}{1+2^3x^2} - \frac{x}{1+x^2}$$

$$U_3(x) = \frac{3^2x}{1+3^3x^2} - \frac{2^2x}{1+2^3x^2}$$

$$U_n(x) = \frac{n^2x}{1+n^3x^2} - \frac{(n-1)^2x}{1+(n-1)^3x^2}$$

$$\therefore\ f_n(x) = U_1(x) + U_2(x) + \cdots + U_n(x) = \frac{n^2x}{1+n^3x^2}$$

$$\therefore\ f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{n^2x}{1+n^3x^2} = \lim_{n\to\infty} \frac{\frac{x}{n}}{\frac{1}{n^3}+x^2}$$

$$= 0 \quad \forall\, x \in [0,1]$$

Now $$|f_n(x) - f(x)| = \left|\frac{n^2x}{1+n^3x^2} - 0\right| = \frac{n^2x}{1+n^3x^2}$$

Let $y = \dfrac{n^2x}{1+n^3x^2}$ $\therefore$ $\dfrac{dy}{dx} = \dfrac{n^2 - n^5x^2}{(1+n^3x^2)^2}$ Solving.

For Max or Min $\dfrac{dy}{dx} = 0 \Rightarrow n^2 - n^5x^2 = 0 \Rightarrow x = \dfrac{1}{n^{3/2}}$

$$\therefore\ \frac{d^2y}{dx^2} = \frac{-2n^3x(3n^2 - n^5x^2)}{(1+n^3x^2)^3} \quad \text{(After solving)}$$

And $$\left.\frac{d^2y}{dx^2}\right]_{x=\frac{1}{n^{3/2}}} = \frac{-2n^3\cdot\frac{1}{n^{3/2}}\left[3n^2 - \frac{n^5}{n^3}\right]}{\left(1+n^3\cdot\frac{1}{n^3}\right)^3} = \frac{-4n^{3/2}}{8}$$

$$= \frac{-n^{3/2}}{2} < 0$$

$\therefore$ y is max. when $x = \dfrac{1}{n^{3/2}}$ and.

$$y_{max} = \frac{n^2 \cdot \frac{1}{n^{3/2}}}{1+n^3 \cdot \frac{1}{n^3}} = \frac{\sqrt{n}}{2}$$

Also $x = \frac{1}{n^{3/2}} \to 0$ as $n \to \infty$

$$\therefore M_n = \sup_{x \in [0,1]} |f_n(x) - f(x)| = \frac{\sqrt{n}}{2}$$

which does not tend to zero as $n \to \infty$

By M_n-Test, $\{f_n\}$ does not converge uniformly on $[0,1]$ ~~and '0' is pt of~~ and hence ΣU_n.

Q. Discuss for uniform Convergence of series

$$\sum_{n=1}^{\infty} \left[\frac{nx}{1+n^2x^2} - \frac{(n+1)x}{1+(n+1)^2x^2} \right] \text{ in } [0,1].$$

Sol^n: Let given series is ΣU_n and $f_n = U_1 + U_2 + \cdots + U_n$ be nth partial sum of ΣU_n.

Here $U_1(x) = \frac{x}{1+1^2x^2} - \frac{2x}{1+2^2x^2}$

$U_2(x) = \frac{2x}{1+2^2x^2} - \frac{3x}{1+3^2x^2}$

$$U_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n+1)x}{1+(n+1)^2x^2}$$

$$\therefore f_n(x) = \frac{x}{1+x^2} - \frac{(n+1)x}{1+(n+1)^2x^2}$$

$$\therefore f(x) = \lim_{n \to \infty} f_n(x) = \begin{cases} \frac{x}{1+x^2} & \text{if } 0 < x \le 1 \\ 0 & \text{if } x = 0 \end{cases}$$

For $0 < x < 1$ and for given $\epsilon > 0$, we have

$$|f_n(x) - f(x)| = \left| \frac{x}{1+x^2} - \frac{(n+1)x}{1+(n+1)^2x^2} - \frac{x}{1+x^2} \right|$$

$$= \frac{(n+1)x}{1+(n+1)^2x^2} < \epsilon$$

if $(n+1)^2 x^2 \epsilon - (n+1) x + \epsilon > 0$

if $(n+1) > \dfrac{x \pm \sqrt{x^2 - 4\epsilon^2 x^2}}{2x^2 \epsilon}$

if $n > -1 + \dfrac{1 + \sqrt{1-4\epsilon^2}}{2x\epsilon}$

Now when $x \to 0$, then $-1 + \dfrac{1+\sqrt{1-4\epsilon^2}}{2x\epsilon} \to \infty$

so it is not possible to choose a the integer m

s.t $|f_n(x) - f(x)| < \epsilon \quad \forall n \geq m$, $\forall x \in [0,1]$

$\therefore$ seq $\{f_n\}$ is not uniformly cgt on $[0,1]$

hence $\sum U_n$.

Q. Test for uniform convergence the series

$$\sum_{n=1}^{\infty} \left[\frac{2n^2x^2}{e^{n^2x^2}} - \frac{2(n-1)^2x^2}{e^{(n-1)^2x^2}} \right]$$

Sol^n^ Let given series is $\sum U_n$ and $f_n = U_1 + U_2 + \cdots + U_n$ be nth partial sum of $\sum U_n$.

$$\therefore \quad U_1(x) = \frac{2x^2}{e^{x^2}} - 0$$

$$U_2(x) = \frac{2 \cdot 2^2 \cdot x^2}{e^{2^2x^2}} - \frac{2 \cdot x^2}{e^{x^2}}$$

$$U_n(x) = \frac{2n^2x^2}{e^{n^2x^2}} - \frac{2(n-1)^2x^2}{e^{(n-1)^2x^2}}$$

Adding, we get

$$f_n(x) = \frac{2n^2x^2}{e^{n^2x^2}}$$

$$\therefore f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{2n^2x^2}{e^{n^2x^2}} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{n\to\infty} \frac{4nx^2}{e^{n^2x^2} \cdot 2nx^2} = \lim_{n\to\infty} \frac{2}{e^{n^2x^2}} = 0 \quad \forall x \in R.$$

$$\therefore |f_n(x) - f(x)| = \left| \frac{2n^2x^2}{e^{n^2x^2}} - 0 \right| = \frac{2n^2x^2}{e^{n^2x^2}}$$

Let $y = \frac{2n^2x^2}{e^{n^2x^2}}$ and $\frac{dy}{dx} = \frac{4n^2x(1-n^2x^2)}{e^{n^2x^2}}$ (Solving)

For Max or Min $\frac{dy}{dx} = 0 \Rightarrow x = \frac{1}{n}$.

Also $\frac{d^2y}{dx^2} = \frac{4n^2(1-5n^2x^2+2n^4x^4)}{e^{n^2x^2}}$ (After Solving)

$$\therefore \left.\frac{d^2y}{dx^2}\right]_{x=\frac{1}{n}} = -\frac{8n^2}{e} < 0$$

$\therefore$ y is max. when $x = \frac{1}{n}$ and

$$y_{max.} = \frac{2}{e}$$

Also $x = \frac{1}{n} \to 0$ as $n \to \infty$

$$\therefore M_n = \sup_{x \in [a,b]} |f_n(x) - f(x)| = \frac{2}{e}$$ [when $[a,b]$ is any interval containing '0']

As $M_n \not\to 0$ as $n \to \infty$

By M_n - Test, $\{f_n\}$ does not converge uniformly on any interval containing '0'.

Hence for ΣU_n.

Q. Test for uniform convergence the series $\sum_{n=0}^{\infty} x e^{-nx}$ in closed interval $[0,1]$.

Sol^n: Let given series is $\sum_{n=0}^{\infty} U_n$ and $f_n = U_0 + U_1 + \cdots + U_{n-1}$ be nth partial sum of $\sum_{n=0}^{\infty} U_n$

$$\therefore f_n(x) = \sum_{n=0}^{n-1} U_n(x) = x + xe^{-x} + \cdots + xe^{-(n-1)x}$$

which is G.P

$$\therefore f_n(x) = \frac{x(1-e^{-nx})}{1-e^{-x}} = \frac{xe^x}{e^x-1}\left(1-\frac{1}{e^{nx}}\right)$$

$$\therefore f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} \frac{xe^x}{e^x-1} & \text{if } 0<x\le 1 \\ 0 & \text{if } x=0 \end{cases}$$

Let $\epsilon>0$ be given and for $0<x\le 1$, we have

$$|f_n(x)-f(x)| = \left|\frac{xe^x}{e^x-1}\left(1-\frac{1}{e^{nx}}\right) - \frac{xe^x}{e^x-1}\right|$$

$$= \left|\frac{-xe^x}{(e^x-1)e^{nx}}\right| = \frac{xe^x}{(e^x-1)e^{nx}} < \epsilon$$

if $\quad \frac{(e^x-1)e^{nx}}{xe^x} > \frac{1}{\epsilon}$

if $\quad \log(e^x-1) + nx - \log x - x > \log\frac{1}{\epsilon}$

if $\quad [\log(e^x-1) - \log x] + nx - x > \log\frac{1}{\epsilon}$

if $\quad [\log(x+\frac{x^2}{\lfloor 2}+\cdots) - \log x] + nx - x > \log\frac{1}{\epsilon}$

if $\quad \log(1+\frac{x}{\lfloor 2}+\cdots) + nx > \log\frac{1}{\epsilon}$

if $\quad n > \frac{\log\frac{1}{\epsilon} + x - \log(1+\frac{x}{\lfloor 2}+\cdots)}{x}$

Now when $x\to 0$, then $\frac{\log\frac{1}{\epsilon}+x-\log(1+\frac{x}{\lfloor 2}+\cdots)}{x} \to \infty$

$\therefore$ It is not possible to choose a the integer m such that $|f_n(x)-f(x)|<\epsilon \quad \forall\, n\ge m,\ \forall\, x\in[0,1]$

$\therefore$ seq $\{f_n\}$ is not uniformly cgt on any interval containing '0', hence Σu_n

Q. Show that series

$(1-x)^2 + (1-x)^2 x + (1-x)^2 x^2 + \dots$ is uniformly cgt on $[0,1]$.

Sol^n Let given series is $\sum U_n$ and $f_n = U_1 + U_2 + \dots + U_n$ be nth partial sum of $\sum U_n$.

$$\therefore f_n(x) = (1-x)^2 [1 + x + x^2 + \dots + x^{n-1}] = (1-x)^2 \frac{(1-x^n)}{1-x}$$

$$= (1-x)(1-x^n)$$

$$\therefore f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} (1-x)(1-x^n)$$

$$= 1-x \qquad \forall x \in [0,1].$$

Now $|f_n(x) - f(x)| = |(1-x)(1-x^n) - (1-x)|$

$$= |(1-x) - x^n(1-x) - (1-x)| = |-x^n(1-x)|$$

$$= x^n (1-x)$$

Let $y = x^n(1-x)$, then $\frac{dy}{dx} = n x^{n-1} - (n+1) x^n$

$$= x^{n-1}[n - (n+1)x]$$

For Max or Min, $\frac{dy}{dx} = 0 \Rightarrow x = \frac{n}{n+1}$

Again $\frac{d^2y}{dx^2} = n(n-1) x^{n-2} - (n+1) n x^{n-1} = n x^{n-2} [(n-1) - (n+1)x]$

$$\therefore \left.\frac{d^2y}{dx^2}\right]_{x=\frac{n}{n+1}} = n\left(\frac{n}{n+1}\right)^{n-2} [n-1-n] = -\frac{n^{n-1}}{(n+1)^{n-2}} < 0 \quad \forall n \in N$$

which show y is max. when $x = \frac{n}{n+1}$

and $y_{max} = \left(\frac{n}{n+1}\right)^n \left[\frac{1}{n+1}\right]$

$$\therefore M_n = \sup_{x\in[0,1]} |f_n(x) - f(x)| = \left(\frac{n}{n+1}\right)^n \left(\frac{1}{n+1}\right)$$

As $M_n \to 0$ as $n \to \infty$

$\therefore$ By M_n test $\{f_n\}$ converge uniformly on $[0,1]$

Q.: If $\sum a_n$ is absolutely cgt, prove that $\sum \frac{a_n x^n}{1+x^{2n}}$ Converges uniformly $\forall x \in R$.

Sol^n: Here $f_n(x) = \dfrac{a_n x^n}{1+x^{2n}}$

Let $y = \dfrac{x^n}{1+x^{2n}}$, then $\dfrac{dy}{dx} = \dfrac{n x^{n-1}(1-x^{2n})}{(1+x^{2n})^2}$ (After Solving)

for max or Min $\dfrac{dy}{dx} = 0 \Rightarrow x = 0, 1, -1$

Also
$$\frac{d^2y}{dx^2} = \frac{(1+x^{2n})^2\left[n(n-1)x^{n-2} - n(3n-1)x^{3n-2}\right] - nx^{n-1}(1-x^{2n})\,2(1+x^{2n})\,2nx^{2n-1}}{(1+x^{2n})^4}$$

$$= \frac{\left[n(n-1)x^{n-2} - n(3n-1)x^{3n-2}\right](1+x^{2n}) - 4n^2x^{3n-2}(1-x^{2n})}{(1+x^{2n})^3}$$

$\therefore \left.\dfrac{d^2y}{dx^2}\right]_{x=1} = \dfrac{-n^2}{2} < 0.$

$\therefore$ y is max. at $x=1$ and $y_{max} = \dfrac{1}{1+1} = \dfrac{1}{2}$

$\therefore$ $|f_n(x)| = \left|\dfrac{a_n x^n}{1+x^{2n}}\right| = \left|\dfrac{x^n}{1+x^{2n}}\right| |a_n| \le \frac{1}{2}|a_n| < |a_n| = M_n$
$\forall x \in R$.

Now $\sum M_n = \sum |a_n|$ is cgt as $\sum a_n$ is absolutely cgt. Hence by W.M Test given series is uniformly cgt $\forall x \in R$.

Q. Show that series $\sum_{n=1}^{\infty} \frac{1}{n^2+x^2}$ is uniformly Cgt on $[0, \infty)$

Sol^n Let given series is $\sum_{n=1}^{\infty} f_n$

where $f_n(x) = \dfrac{1}{n^2+x^2}$, $x \in [0, \infty)$

$\therefore |f_n(x)| = \left|\frac{1}{n^2+x^2}\right| = \frac{1}{n^2+x^2}$

$\leq \frac{1}{n^2} = M_n \quad \forall x \in [0,\infty)$

$\left[\begin{array}{l} \text{for } x \in [0,\infty) \\ x^2 \geq 0 \\ \Rightarrow x^2+n^2 \geq n^2 \\ \Rightarrow \frac{1}{x^2+n^2} \leq \frac{1}{n^2} \end{array}\right]$

Now $\sum M_n = \sum \frac{1}{n^2}$ is cgt by p-test

$\therefore$ By W.M. Test $\sum f_n$ is uniformly cgt on $[0,\infty)$.

Q Show that series $\sum_{n=1}^{\infty} \frac{x}{n(n+1)}$ is uniformly cgt in $[0,k]$, $k>0$.

Solⁿ Let given series is $\sum_{n=1}^{\infty} f_n$, where

$f_n(x) = \frac{x}{n(n+1)}, \quad x \in [0,k]$.

$\therefore |f_n(x)| = \left|\frac{x}{n(n+1)}\right| = \frac{x}{n(n+1)} \leq \frac{k}{n(n+1)} < \frac{k}{n^2} = M_n$

$\forall x \in [0,k]$
$\forall n \in N$.

Now $\sum M_n = k\sum \frac{1}{n^2}$ is cgt by p-test

$\therefore$ By W-M test $\sum f_n$ is uniformly cgt on $[0,k]$.

Q. Test the convergence of series $\sum \frac{1}{(x^2+n)(x^2+nx)}$

Solⁿ: Let given series is $\sum f_n$, where

$f_n(x) = \frac{1}{(x^2+n)(x^2+nx)}$

$|f_n(x)| = \left|\frac{1}{(x^2+n)(x^2+nx)}\right| = \frac{1}{(x^2+n)(x^2+nx)} \quad \forall x \geq 1$

$< \frac{1}{n^2} \quad \forall x \geq 1, \ \forall n \in N$

$= M_n$

for $x \geq 1 > 0$
$x^2 > 0$
$\Rightarrow x^2+n > n$ and $x^2+nx > nx \geq n$
$\Rightarrow \frac{1}{x^2+n} < \frac{1}{n}$ and $\frac{1}{x^2+nx} \leq \frac{1}{n}$
$\Rightarrow \frac{1}{(x^2+n)(x^2+nx)} < \frac{1}{n^2}$

Now $\sum M_n = \sum \frac{1}{n^2}$ is cgt by p-test

$\therefore$ By W.M test Σf_n is uniformly cgt $\forall x \geq 1$

Q. Show that $\sum_{n=1}^{\infty} \frac{1}{(x^2+n)(x^2+n+1)}$ is uniformly cgt $\forall x \in R$

Sol^n: Let given series is $\sum_{n=1}^{\infty} f_n$, where

$$f_n(x) = \frac{1}{(x^2+n)(x^2+n+1)}, \quad x \in R.$$

$$\therefore |f_n(x)| = \left|\frac{1}{(x^2+n)(x^2+n+1)}\right| = \frac{1}{(x^2+n)(x^2+n+1)}$$

$< \frac{1}{n^2}$ $\forall n \in N$, $\forall x \in R$

$= M_n$

$\left[\begin{array}{l} \forall x \in R \\ x^2 \geq 0 \\ \Rightarrow x^2+n \geq n \\ \text{and } x^2+n+1 \geq n+1 > n \\ \Rightarrow \frac{1}{x^2+n} \leq \frac{1}{n} \text{ and} \\ \frac{1}{x^2+n+1} < \frac{1}{n} \\ \Rightarrow \frac{1}{(x^2+n)(x^2+n+1)} < \frac{1}{n^2} \end{array}\right]$

Now $\Sigma M_n = \Sigma \frac{1}{n^2}$ is cgt by p-Test

$\therefore$ By W-M test Σf_n is uniformly cgt $\forall x \in R$

Q. Show that series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^p}$ is uniformly and absolutely cgt $\forall x \in R$ and $p > 1$.

Sol^n: Let given series is Σf_n, where

$$f_n(x) = \frac{\sin nx}{n^p}, \quad x \in R, \; p > 1$$

$$\therefore |f_n(x)| = \left|\frac{\sin nx}{n^p}\right| = \frac{|\sin nx|}{n^p} \leq \frac{1}{n^p} = M_n \quad \forall n \in N, \forall x \in R$$

Now $\Sigma M_n = \Sigma \frac{1}{n^p}$ is cgt by p-test as $p > 1$. $\left[\because |\sin \theta| \leq 1 \; \forall \theta \in R\right]$

$\therefore$ By W-M Test Σf_n is uniformly cgt $\forall x \in R$ and $p > 1$.

Also $\Sigma |f_n(x)| = \Sigma \frac{|\sin nx|}{n^p} = \Sigma u_n(x)$

where $u_n(x) = \frac{|\sin nx|}{n^p}$

$\therefore\ |U_n(x)| = \left|\frac{|\sin nx|}{n^p}\right| = \frac{|\sin nx|}{n^p} \le \frac{1}{n^p} = M_n' \quad \forall n \in N,\ \forall x \in R$

Now $\Sigma M_n' = \Sigma \frac{1}{n^p}$ is cgt by p-test as $p>1$

$\therefore$ ~~$\Sigma U_n(x)$~~ By Wl-M-Test

$\Sigma U_n(x) = \Sigma |f_n(x)|$ is cgt $\Rightarrow \Sigma f_n(x)$ is absolutely cgt $\forall x \in R$

Q. Show that $\Sigma \frac{\cos n\theta}{n^p}$ is uniformly and absolutely cgt $\forall\ \theta \in R$, $p>1$.

Sol^n : Same as previous question.

Q. Show that following series are uniformly cgt $\forall x \in R$ (i) $\sum_{n=1}^{\infty} \frac{\sin(x^2+n^2x)}{n(n+2)}$ (ii) $\sum_{n=1}^{\infty} \frac{\cos(x^2+n^2x)}{n(n^2+2)}$

Sol^n (i) Let given series is Σf_n, where

$$f_n(x) = \frac{\sin(x^2+n^2x)}{n(n+2)}, \quad x \in R$$

$\therefore\ |f_n(x)| = \left|\frac{\sin(x^2+n^2x)}{n(n+2)}\right| = \frac{|\sin(x^2+n^2x)|}{n(n+2)} \le \frac{1}{n(n+2)} < \frac{1}{n^2} = M_n \quad \forall n \in N,\ x \in R$

Now $\Sigma M_n = \Sigma \frac{1}{n^2}$ is cgt by p-test $\left[\because |\sin\theta| \le 1\ \forall\ \theta \in R.\right.$

$\therefore$ By Wl-M-Test, Σf_n is uniformly cgt $\forall x \in R$.

(ii) Similar proof as part (i)

Q. Show that series $\sum r^n \cos n\theta$, $0<r<1$ Converges uniformly $\forall\ \theta \in R$.

Sol^n Let given series is $\sum f_n$, where

$$f_n(\theta) = r^n \cos n\theta \quad , \quad \theta \in R.$$

$$\therefore |f_n(\theta)| = |r^n \cos n\theta| = r^n |\cos n\theta| \le r^n = M_n \quad \forall\ n \in N,\ \theta \in R.$$

$[\because |\cos n\theta| \le 1 \ \forall\ \theta \in R]$

Now $\sum M_n = \sum r^n$ is G.P series with c.R r, $0<r<1$

$\therefore \sum M_n$ is cgt

By W-M test, $\sum f_n(\theta)$ is uniformly cgt $\forall\ \theta \in R$

Q. Show that if $0<r<1$, then series $\sum\limits_{n=1}^{\infty} r^n \sin a^n x$ is uniformly cgt on R.

Sol^n: Same as previous question

Q. If the series $\sum a_n$ converges absolutely, then prove that $\sum a_n \cos nx$ and $\sum a_n \sin nx$ are uniformly cgt on R.

Sol^n Let given series is $\sum f_n(x)$, where

$$f_n(x) = a_n \cos nx \quad , \quad x \in R.$$

$$\therefore |f_n(x)| = |a_n \cos nx| = |a_n||\cos nx| \le |a_n| = M_n \quad \forall\ x \in R,\ \forall\ n \in N$$

Now $\sum M_n = \sum |a_n|$ is cgt $\quad [\because \sum a_n$ is absolutely cgt $\Rightarrow \sum |a_n|$ is cgt$]$

By W-M Test, $\sum f_n(x)$ is uniformly cgt $\forall\ x \in R$.

IIIly we can prove $\sum a_n \cos nx$ is uniformly cgt $\forall\ x \in R$

Q. Show that series $\sum \frac{x^n}{1+x^n}$ Converges uniformly on $[0,a]$, $0<a<1$.

Sol^n: Let given series $\sum f_n$, where

$$f_n(x) = \frac{x^n}{1+x^n}, \quad x \in [0,a],\ 0<a<1.$$

$$\therefore |f_n(x)| = \left|\frac{x^n}{1+x^n}\right| = \frac{x^n}{1+x^n} \le x^n \le a^n = M_n \quad \forall x \in [0,a]\ \forall n \in N$$

$\left[\ x^n \ge 0 \Rightarrow 1+x^n \ge 1 \Rightarrow \frac{1}{1+x^n} \le 1 \Rightarrow \frac{x^n}{1+x^n} \le x^n\right.$

Now $\sum M_n = \sum a^n$ is G.P series with C.R. a, $0<a<1$

$\therefore \sum M_n$ is cgt

By W-M test, series $\sum f_n(x)$ is uniformly cgt $\forall x \in [0,a]$, $0<a<1$

Q. Discuss the uniform Convergence of $\sum \frac{x}{(n+x^2)^2}$

Sol^n Let given series is $\sum f_n$

where $f_n(x) = \frac{x}{(n+x^2)^2}$

Let $y = \frac{x}{(n+x^2)^2} \quad \therefore \frac{dy}{dx} = \frac{n-3x^2}{(n+x^2)^3}$ (Solving)

$$\frac{d^2y}{dx^2} = \frac{(n+x^2)^3(-6x) - (n-3x^2)3(n+x^2)^2 2x}{(n+x^2)^6} = \frac{-6x[n+x^2+n-3x^2]}{(n+x^2)^4}$$

$$= \frac{-6x(2n-2x^2)}{(n+x^2)^4}.$$

for max or min $\frac{dy}{dx} = 0 \Rightarrow n-3x^2 = 0 \Rightarrow x = \sqrt{\frac{n}{3}}$

$$\therefore \left.\frac{d^2y}{dx^2}\right]_{x=\sqrt{\frac{n}{3}}} = \frac{-27\sqrt{3}}{32\, n^{5/2}} < 0.$$

$\therefore y$ is max at $x = \sqrt{\frac{n}{3}}$

and $y_{max} = \frac{\sqrt{\frac{n}{3}}}{(n+\frac{n}{3})^2} = \frac{3\sqrt{3}}{16} \times \frac{1}{n^{3/2}}$

Now $|f_n(x)| = \left|\frac{x}{(n+x^2)^2}\right| \le \frac{3\sqrt{3}}{16} \times \frac{1}{n^{3/2}} = M_n \quad \forall n \in N\ \forall x \in R.$